Woche 12

06.01.2011

1. Inhomogene Flächen

Bestimmen Sie Masse, Schwerpunkt und Trägheitsmomente der auf dem Intervall $[0,2]$ durch die Funktionen

$$f_o(x) := 1 \ , \qquad f_u(x) := 0$$

definierten Fläche für die Dichten

$$\rho_1(x,y) := xy^2 \ , \qquad \rho_2(x,y) := x \ , \qquad \rho_3(x,y) := y^2 \ .$$

(a) $\rho_1(x,y) = xy^2$: Es gilt

$$\begin{aligned} m &= \int_F \rho_1(x,y)\ dF = \int_0^2 \int_0^1 xy^2\ dy\ dx = \int_0^2 [\frac{1}{3}xy^3]_0^1\ dx \\ &= \int_0^2 \frac{1}{3}x\ dx = [\frac{1}{6}x^2]_0^2 = \frac{4}{6} = \frac{2}{3} \ , \end{aligned}$$

$$\begin{aligned} x_S &= \int_F x \cdot \rho_1(x,y)\ dF = \int_0^2 \int_0^1 x^2y^2\ dy\ dx = \int_0^2 [\frac{1}{3}x^2y^3]_0^1\ dx \\ &= \int_0^2 \frac{1}{3}x^2\ dx = [\frac{1}{9}x^3]_0^2 = \frac{8}{9} \ , \\ y_S &= \int_F y \cdot \rho_1(x,y)\ dF = \int_0^2 \int_0^1 xy^3\ dy\ dx = \int_0^2 [\frac{1}{4}xy^4]_0^1\ dx \\ &= \int_0^2 \frac{1}{4}x\ dx = [\frac{1}{8}x^2]_0^2 = \frac{4}{8} = \frac{1}{2} \end{aligned}$$

und

$$\begin{aligned} I_y &= \int_F x^2 \cdot \rho_1(x,y)\ dF = \int_0^2 \int_0^1 x^3y^2\ dy\ dx = \int_0^2 [\frac{1}{3}x^3y^3]_0^1\ dx \\ &= \int_0^2 \frac{1}{3}x^3\ dx = [\frac{1}{12}x^4]_0^2 = \frac{16}{12} = \frac{4}{3} \ , \\ I_x &= \int_F y^2 \cdot \rho_1(x,y)\ dF = \int_0^2 \int_0^1 xy^4\ dy\ dx = \int_0^2 [\frac{1}{5}xy^5]_0^1\ dx \\ &= \int_0^2 \frac{1}{5}x\ dx = [\frac{1}{10}x^2]_0^2 = \frac{4}{10} = \frac{2}{5} \ , \end{aligned}$$

Achtung: Es gilt

$$\int_0^2 \rho_1(x,y)(f_o(x) - f_u(x))\ dx = \int_0^2 xy^2\ dx = [\frac{1}{2}x^2y^2]_0^2 = 2y^2 \neq m$$

und

$$\int_0^2 \rho_1(x,y) \cdot x \cdot (f_o(x) - f_u(x))\ dx = \int_0^2 x^2y^2\ dx = [\frac{1}{3}x^3y^2]_0^2 = \frac{8}{3}y^2 \neq x_S$$

etc.

(b) $\rho_2(x,y) = x$: Es gilt

$$m = \int_F \rho_2(x,y)\ dF = \int_0^2 \int_0^1 x\ dy\ dx = \int_0^2 [xy]_0^1\ dx$$
$$= \int_0^2 x\ dx = [\frac{1}{2}x^2]_0^2 = \frac{4}{2} = 2\ ,$$

$$x_S = \int_F x \cdot \rho_2(x,y)\ dF = \int_0^2 \int_0^1 x^2\ dy\ dx = \int_0^2 [x^2y]_0^1\ dx$$
$$= \int_0^2 x^2\ dx = [\frac{1}{3}x^3]_0^2 = \frac{8}{3}\ ,$$
$$y_S = \int_F y \cdot \rho_2(x,y)\ dF = \int_0^2 \int_0^1 xy\ dy\ dx = \int_0^2 [\frac{1}{2}xy^2]_0^1\ dx$$
$$= \int_0^2 \frac{1}{2}x\ dx = [\frac{1}{4}x^2]_0^2 = \frac{4}{4} = 1$$

und

$$I_y = \int_F x^2 \cdot \rho_2(x,y)\ dF = \int_0^2 \int_0^1 x^3\ dy\ dx = \int_0^2 [x^3y]_0^1\ dx$$
$$= \int_0^2 x^3\ dx = [\frac{1}{4}x^4]_0^2 = \frac{16}{4} = 4\ ,$$
$$I_x = \int_F y^2 \cdot \rho_2(x,y)\ dF = \int_0^2 \int_0^1 xy^2\ dy\ dx = \int_0^2 [\frac{1}{3}xy^3]_0^1\ dx$$
$$= \int_0^2 \frac{1}{3}x\ dx = [\frac{1}{6}x^2]_0^2 = \frac{4}{6} = \frac{2}{3}\ .$$

Achtung: In diesem Fall gilt

$$\int_0^2 \rho_2(x,y) \cdot (f_o(x) - f_u(x))\ dx = \int_0^2 x\ dx = [\frac{1}{2}x^2]_0^2 = 2 = m$$

und

$$\int_0^2 \rho_2(x,y) \cdot x \cdot (f_o(x) - f_u(x))\ dx = \int_0^2 x^2\ dx = [\frac{1}{3}x^3]_0^2 = \frac{8}{3} = x_S$$

etc. Dies funktioniert, weil die Dichte nur von x abhängt.

(c) $\rho_3(x, y) = y^2$: Es gilt

$$m = \int_F \rho_3(x, y)\ dF = \int_0^2 \int_0^1 y^2\ dy\ dx = \int_0^2 [\frac{1}{3}y^3]_0^1\ dx$$
$$= \int_0^2 \frac{1}{3}\ dx = [\frac{1}{3}x]_0^2 = \frac{2}{3}\ ,$$

$$x_S = \int_F x \cdot \rho_3(x, y)\ dF = \int_0^2 \int_0^1 xy^2\ dy\ dx = \int_0^2 [\frac{1}{3}xy^3]_0^1\ dx$$
$$= \int_0^2 \frac{1}{3}x\ dx = [\frac{1}{6}x^2]_0^2 = \frac{4}{6} = \frac{2}{3}\ ,$$
$$y_S = \int_F y \cdot \rho_1(x, y)\ dF = \int_0^2 \int_0^1 y^3\ dy\ dx = \int_0^2 [\frac{1}{4}y^4]_0^1\ dx$$
$$= \int_0^2 \frac{1}{4}\ dx = [\frac{1}{4}x]_0^2 = \frac{2}{4} = \frac{1}{2}$$

und

$$I_y = \int_F x^2 \cdot \rho_3(x, y)\ dF = \int_0^2 \int_0^1 x^2y^2\ dy\ dx = \int_0^2 [\frac{1}{3}x^2y^3]_0^1\ dx$$
$$= \int_0^2 \frac{1}{3}x^2\ dx = [\frac{1}{9}x^3]_0^2 = \frac{8}{9}\ ,$$
$$I_x = \int_F y^2 \cdot \rho_1(x, y)\ dF = \int_0^2 \int_0^1 y^4\ dy\ dx = \int_0^2 [\frac{1}{5}y^5]_0^1\ dx$$
$$= \int_0^2 \frac{1}{5}\ dx = [\frac{1}{5}x]_0^2 = \frac{2}{5}\ .$$

Achtung: In diesem Fall gilt

$$\int_0^2 \rho_3(x, y)(f_o(x) - f_u(x))\ dx = \int_0^2 y^2\ dx = 2y^2 \neq m$$

und

$$\int_0^2 \rho_3(x, y) \cdot x \cdot (f_o(x) - f_u(x))\ dx = \int_0^2 y^2x\ dx = [\frac{1}{2}y^2x^2]_0^2 = 2y^2 \neq x_S$$

etc. Definiert man die Fläche allerdings über die Funktionen

$$g_o(y) := 2\ , \qquad g_u(y) := 0$$

auf dem Intervall $[0, 1]$, so gilt

$$\int_0^1 \rho_3(x, y)(g_o(y) - g_u(y))\ dy = \int_0^1 2y^2\ dy = [\frac{2}{3}y^3]_0^1 = \frac{2}{3} = m$$

und

$$\frac{1}{2}\int_0^1 \rho_3(x, y)((g_o(y))^2 - (g_u(y))^2)\ dy = \frac{1}{2}\int_0^1 4y^2\ dy = 2[\frac{1}{3}y^3]_0^1 = \frac{2}{3} = x_S$$

etc.

2. Aufgaben für den 13.01.11

(a) Mehrdimensionale Integration: Bestimmen Sie die Integralformeln für Masse, Schwerpunkt und Hauptträgheitsmomente des durch

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) := x^2 + y^2$$

definierten Paraboloids

$$V := \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y) \leq z \leq h\}$$

der Höhe h in

(i) kartesischen Koordinaten

(ii) Zylinderkoordinaten

bzgl. aller Integrationsreihenfolgen und der Dichte $\rho = \rho(x,y,z) = \rho(r,\varphi,z)$.